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Relaxation-Time Ansatz for Quantum Transport Theory: Spin Effects*

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A relaxation-time ansatz which treats both orbital and spin relaxation in quantum electron transport theory is presented. For orbital relaxation we use an ansatz which conserves both charge and spin density, and which is a modification of the treatment in which the collision processes are assumed to relax the system to a state of instantaneous local thermal equilibrium. For spin relaxation the ansatz conserves only the charge density. The general method yields gauge-invariant charge-conserving results for the response of the system to an electromagnetic perturbation, and yields physically correct results for the spin resonance response of the electronic system. General results are derived for the linear-response part of the one-electron density matrix for the case of a space- and time-varying perturbation corresponding to a single Fourier component. The results are applied to the calculation of the electrical conductivity, the magnetic susceptibility, and the cross section for inelastic light scattering from semiconductor magnetoplasmas.

I. INTRODUCTION

In many calculations of quantum transport processes, a useful first approximation is to treat the effect of collisions on the carriers by introducing a phenomenological relaxation-time term into the transport equation. If the relaxation time τ is measured in one experiment, and if it does not vary rapidly with frequency, then the value obtained can be applied to analyze other experimental situations. Though simple in essence, the use of a relaxation-time ansatz in quantum transport theory is not without its hazards. Naive forms of the ansatz fail (a) to conserve charge and/or (b) to give gauge-invariant results. This latter difficulty is circumvented in classical or semiclassical treatments because the transport equation is formulated directly in terms of the electromagnetic fields rather than the corresponding vector and scalar potentials.

It has been recognized for some time now that the simplest physical assumption, i.e., that the collision processes should tend to relax the system to its static equilibrium state, leads to unphysical results in a wide variety of dynamical problems. A modification which appears to eliminate most of these difficulties is based, instead, on the assumption that the collision processes will relax the system to an equilibrium state determined by the total instantaneous Hamiltonian.[1-7] However, it has also been recognized, in the case of response to a general electromagnetic perturbation, that this second approach does not yield gauge-invariant results.[5] Basically, this problem can be traced to the fact that neither of the above methods satisfies the condition of charge conservation when the response of the system involves longitudinal density fluctuations.

Several semiclassical[8-10] transport treatments have been presented which eliminate this difficulty by introducing a local chemical potential $\mu(\vec{r}, t)$ which is determined in such a way that charge conservation is satisfied. An extension of this method to the quantum case was suggested by Tosima, Quinn, and Lampert[11] and has been carried out for the general conductivity problem by Mertsching,[12] and Greene, Lee, Quinn, and Rodriquez.[13] In addition, the case of purely longitudinal disturbances has recently been discussed by Mermin,[14] who has corrected the conjectured results proposed by Kliewer and Fuchs.[15] We shall refer to the above method of treating relaxation effects as relaxation to instantaneous local equilibrium (RILE).

The essence of the RILE method is to represent the effect of collisions in the equation of motion of the one-electron density matrix by a relaxation term of the form

$$\left(\frac{d\rho}{dt}\right)_c = -\frac{\rho-\bar{\rho}}{\tau} , \qquad (1)$$

where $(d\rho/dt)_c$ is the rate of change of ρ due to collisions, and

$$\bar{\rho} = \rho_0(H(t),\ \mu(\vec{r}, t)) . \qquad (2)$$

Here ρ_0 denotes the Fermi function, $H(t)$ is the total instantaneous Hamiltonian in the absence of collisions, and $\mu(\vec{r}, t)$ is the local chemical potential which is determined by requiring conservation of charge in the relaxation processes. The form of Eq. (2) can be simplified for certain special cases. For purely transverse[11] disturbances, μ can be replaced by μ_0, the equilibrium chemical potential, because there is no induced charge density. For the case of purely longitudinal disturbances,[14] where one can introduce the perturbation using only a scalar potential, H can be replaced by the time-independent Hamiltonian H_0 because the condition for charge conservation determines only the *difference* between the chemical and scalar potentials. However for the general conductivity problem, both $H(t)$ and $\mu(\vec{r}, t)$ must be included[12,13] in Eq. (2) to give charge-conserving gauge-invariant results.

Although the RILE method described above is satisfactory for treating the problem of electrical conductivity, it is inadequate for treating transport problems in which spin density perturbations are important. The purpose of this article is to give a relaxation-time ansatz for quantum transport theory which is capable of treating spin effects, and includes both an orbital and a spin relaxation time.

In many physical situations in solids, there are two important and quite distinct relaxation processes: (a) rapid momentum randomizing collisions,[16] and (b) slower spin relaxation effects.[17] The former often involve electron collisions with phonons and impurities.[16] We take a simple model for these collisions, assuming an isotropic relaxation time τ_p, and assuming that their main effect is randomization of the momentum. The charge density and spin density of the carriers are assumed to be unaffected by these processes. The spin relaxation processes destroy spin density perturbations, but conserve the charge density. We assume an isotropic spin relaxation time τ_s, and an additive ansatz of the form

$$\left(\frac{d\rho}{dt}\right)_c = -\frac{\rho-\bar{\rho}_p}{\tau_p} - \frac{\rho-\bar{\rho}_s}{\tau_s} . \qquad (3)$$

We have in mind throughout the paper physical situations such that $\tau_s \gg \tau_p$, a convenient but not generally essential condition.

The crucial point in Eq. (3) is the choice of $\bar{\rho}_p$ and $\bar{\rho}_s$. Since we shall address ourselves to problems in which charge and spin density perturbations are important, we concentrate on these two quantities, requiring the momentum relaxation processes to conserve the charge density

$$\operatorname{tr}[e\,\delta(\vec{x}-\vec{r})\,\rho] = \operatorname{tr}[e\,\delta(\vec{x}-\vec{r})\bar{\rho}_p] \qquad (4)$$

and the spin density

$$\operatorname{tr}[\vec{\sigma}\delta(\vec{x}-\vec{r})\rho] = \operatorname{tr}[\vec{\sigma}\delta(\vec{x}-\vec{r})\bar{\rho}_p] . \qquad (5)$$

By definition, the spin relaxation processes do not conserve spin density, but we require that they do conserve the charge density

$$\operatorname{tr}[e\,\delta(\vec{x}-\vec{r})\rho] = \operatorname{tr}[e\,\delta(\vec{x}-\vec{r})\,\bar{\rho}_s] . \qquad (6)$$

The RILE method employed in Refs. 12–14 satisfies Eq. (4) but not Eq. (5). In this paper we choose $\bar{\rho}_p$ by modifying the RILE approach so that both

Eqs. (4) and (5) are satisfied. We choose $\bar{\rho}_s = \rho_0[H(t), \mu(\vec{r}, t)]$ in accordance with the ordinary RILE method so that Eq. (6) is satisfied. In our treatment, we ignore spin-orbit coupling to the extent that we assume that the one-electron eigenstates are well represented by single-term products of an orbital and spin part. The above discussion and Eqs. (3)–(6) are quite analogous to their counterparts which are found in quasiclassical treatments[18, 19] of electron spin resonance in metals, which use the Boltzmann transport equation.

In Sec. II of this paper, we consider the linear part of the one-electron density matrix which develops in response to a time- and space-dependent perturbation corresponding to a single Fourier component. We modify the ordinary RILE treatment of the orbital relaxation term to give spin conservation for these processes. The general result for the density matrix elements from this approach is extremely complicated and is given in Appendix A. To facilitate example calculations carried out in Secs. III and IV, we present in Sec. II the density matrix results which obtain using the simpler ansatz $\bar{\rho}_s = \rho_0(H_0, \mu_0)$. For $\tau_s \gg \tau_p$ this choice is expected to be reasonable for situations such that (a) the off-resonant perpendicular magnetization is unimportant and (b) $\omega\tau_s \gg 1$. This latter condition ensures that the electromagnetic response which obtains using the simpler ansatz for the spin relaxation term is approximately gauge invariant. The simplification gained by this approximation is considerable, particularly for the light-scattering calculation of Sec. IV.

In Sec. III, we apply the density matrix results to two relatively simple problems. First, we consider the current response to an electric field. We calculate the conductivity tensor and the longitudinal dielectric function, and compare our results with those derived previously. Second, we calculate the magnetization induced by a time-dependent B field. In the long-wavelength limit, the poles of the susceptibility are found to be broadened *only* by τ_s, as would be expected physically. This is in contrast to the results one obtains if the orbital relaxation ansatz does not conserve spin density. Finally, in Sec. IV we use our ansatz to include phenomenological damping in a calculation of the cross section for inelastic light scattering from semiconductor plasmas in a magnetic field.[20] Light-scattering experiments can couple to a variety of orbital and spin excitations,[21] and the results of this section are an important application of our ansatz which includes both orbital and spin relaxation effects. In a forthcoming paper,[22] we use the results of Sec. IV in a detailed study of magnetoplasma quasi-elastic scattering from acoustic plasma waves and single-electron excitations in semiconductors.

II. RELAXATION-TIME ANSATZ

We take the single-particle density matrix to satisfy the equation of motion

$$i\hbar \frac{d\rho}{dt} = [H, \rho] - \frac{i\hbar}{\tau_p}[\rho - \bar{\rho}_p] - \frac{i\hbar}{\tau_s}[\rho - \bar{\rho}_s] , \qquad (7)$$

in accordance with our discussion of Eq. (3) in Sec. I. Here $H(t) = H_0 + \mathcal{V}'(t)$, with $\mathcal{V}'(t)$ a general time-dependent perturbation of the form $\mathcal{V}'(t) = \mathcal{V}(\vec{r}, \vec{p}, \vec{\sigma}, \omega)e^{i\omega t}$, where we assume the $\vec{r}$ dependence involves a single Fourier component proportional to $e^{-i\vec{q}\cdot\vec{r}}$. The eigenstates of the single-particle Hamiltonian H_0 are denoted by $|\alpha\rangle$ with $H_0|\alpha\rangle = \epsilon_\alpha|\alpha\rangle$. Where appropriate, H_0 is assumed to contain an interaction with a uniform magnetic field $\vec{B}_0$. We begin here by following the RILE method and requiring both relaxation terms in Eq. (7) to conserve charge density. This result will then be modified so that the orbital relaxation term satisfies the spin conservation condition [Eq. (5)].

We linearize Eq. (7) by defining

$$\rho = \rho_0(H_0, \mu_0) + \rho' e^{i\omega t}$$

and

$$\mu(\vec{r}, t) = \mu_0 + \delta\mu\, e^{i(\omega t - \vec{q}\cdot\vec{r})} .$$

We expand all quantities (including $\bar{\rho}_p$ and $\bar{\rho}_s$) to first order in $\mathcal{V}'(t)$. This scheme of linearization is slightly different than the one employed in Ref. 13, but it may be shown to give identical results for various average values of physical interest. Determining $\delta\mu$ such that the τ_p, τ_s terms *individually* conserve charge, we obtain as the linearized equation of motion

$$(\hbar\omega + \epsilon_\alpha - \epsilon_\beta)\rho'_{\alpha\beta} = \mathcal{V}_{\alpha\beta}[\rho_0(\epsilon_\alpha) - \rho_0(\epsilon_\beta)] + \frac{i\hbar}{\tau_p}[\rho'_{\alpha\beta} - (\bar{\rho}'_p)_{\alpha\beta}] + \frac{i\hbar}{\tau_s}[\rho'_{\alpha\beta} - (\bar{\rho}'_s)_{\alpha\beta}] , \qquad (8)$$

with

$$(\bar{\rho}'_p)_{\alpha\beta} = (\bar{\rho}'_s)_{\alpha\beta} = \bar{\rho}'_{\alpha\beta}$$

and

$$\bar{\rho}'_{\alpha\beta} = \mathcal{V}_{\alpha\beta}\Lambda^{(2)}_{\alpha\beta} + n^*_{\beta\alpha}\Lambda^{(2)}_{\alpha\beta} \times \left[\sum_{\gamma\delta} n_{\delta\gamma}(\rho'_{\gamma\delta} - \mathcal{V}_{\gamma\delta}\Lambda^{(2)}_{\gamma\delta}) \Big/ \sum_{\gamma\delta} |n_{\delta\gamma}|^2 \Lambda^{(2)}_{\gamma\delta}\right] , \qquad (9)$$

$$n_{\alpha\beta} = \langle\alpha|n(\vec{q})|\beta\rangle = \langle\alpha|e^{i\vec{q}\cdot\vec{r}}|\beta\rangle , \qquad (10)$$

and

$$\Lambda^{(2)}_{\alpha\beta} = \frac{\rho_0(\epsilon_\alpha) - \rho_0(\epsilon_\beta)}{\epsilon_\alpha - \epsilon_\beta} . \qquad (11)$$

For later purposes, we also define

$$\Lambda^{(1)}_{\alpha\beta} = \frac{\rho_0(\epsilon_\alpha) - \rho_0(\epsilon_\beta)}{\epsilon_\alpha - \epsilon_\beta + \hbar(\omega - i/\tau)} , \qquad (12)$$

where

$$(1/\tau)=(1/\tau_p)+(1/\tau_s)\ .$$

At this point our results satisfy the charge-conservation conditions [Eqs. (4) and (6)], but are unphysical in that τ_p and τ_s play exactly the same role. We could, in fact, combine the relaxation terms of Eq. (8) into a single term with an effective relaxation time $1/\tau=1/\tau_p+1/\tau_s$. Because $\tau_s \gg \tau_p$ for many materials, the spin relaxation term would become unimportant. We now consider single-particle eigenstates whose wave functions are well approximated by a single-term product of a space and spin part, i.e., $|\alpha\rangle=|\mu\rangle\,|\sigma\rangle$, where μ denotes orbital quantum numbers and σ denotes spin quantum numbers. For the spin relaxation term we take over Eq. (9) without modification:

$$(\overline{\rho}_s')_{\mu\sigma,\nu\sigma'}=\mathcal{V}_{\mu\sigma,\nu\sigma'}\Lambda^{(2)}_{\mu\sigma,\nu\sigma'}+n^*_{\nu\mu}\,\delta_{\sigma,\sigma'}\Lambda^{(2)}_{\mu\sigma,\nu\sigma}\times\left(\sum_{\substack{\kappa\lambda\\ \sigma''}} n_{\lambda\kappa}(\rho'_{\kappa\sigma'',\lambda\sigma''}-\mathcal{V}_{\kappa\sigma'',\lambda\sigma''}\Lambda^{(2)}_{\kappa\sigma'',\lambda\sigma''})\Big/\sum_{\substack{\kappa\lambda\\ \sigma''}}|n_{\lambda\kappa}|^2\Lambda_{\kappa\sigma'',\lambda\sigma''}\right). \tag{13}$$

For the orbital relaxation term we modify Eq. (13) to read

$$(\overline{\rho}_p')_{\mu\sigma,\nu\sigma'}=\mathcal{V}_{\mu\sigma,\nu\sigma'}\Lambda^{(2)}_{\mu\sigma,\nu\sigma'}+n^*_{\gamma\nu}\Lambda^{(2)}_{\mu\sigma,\nu\sigma'}\times\left(\sum_{\kappa\lambda} n_{\lambda\kappa}(\rho'_{\kappa\sigma,\lambda\sigma'}-\mathcal{V}_{\kappa\sigma,\lambda\sigma'}\Lambda^{(2)}_{\kappa\sigma,\lambda\sigma'})\Big/\sum_{\kappa\lambda}|n_{\lambda\kappa}|^2\Lambda^{(2)}_{\kappa\sigma,\lambda\sigma'}\right). \tag{14}$$

Note that only the orbital quantum numbers are summed in Eq. (14). By direct computation one can now see that Eq. (14) satisfies both the charge- and spin-conservation conditions [Eqs. (4) and (5)]. In fact, Eq. (14) satisfies the more general condition $\mathrm{Tr}[f(\vec{\sigma})e^{i\vec{q}\cdot\vec{r}}\rho]=\mathrm{Tr}[f(\vec{\sigma})e^{i\vec{q}\cdot\vec{r}}\overline{\rho}_p]$, where $f(\vec{\sigma})$ is either the identity operator or any function of electron spin. This more general relation[23] would be expected to hold for any scattering operator which is independent of $\vec{\sigma}$ and $\vec{p}$.

The general solution of Eq. (8) with the collision terms determined from Eqs. (13) and (14) is complicated and is presented in Appendix A. However, for many problems of physical interest it is sufficient to set $\overline{\rho}_s=\rho_0(H_0,\mu_0)$ ($\overline{\rho}_s'=0$), which corresponds to spin relaxation to the static equilibrium state. The limitations of this approximation are discussed in detail in Appendix A. There it is shown that when $\tau_s\gg\tau_p$, spin relaxation terms play no essential role in the electrical-response problem and can be dropped altogether. Even if one were to include τ_s in the conductivity problem using the approximate ansatz above, the results one obtains are approximately gauge invariant as long as the condition $\omega\tau_s\gg 1$ is satisfied. In the electron-spin-resonance problem, the simpler treatment ($\overline{\rho}_s'=0$) correctly yields the resonant component of the perpendicular spin susceptibility, but is inadequate for describing the nonresonant response. However, in this problem we can show that, with the orbital relaxation processes correctly treated in a manner which conserves spin density, τ_p cancels out of the final result for the long-wavelength perpendicular susceptibility. Thus, a simplified treatment of this problem could be obtained by ignoring the orbital relaxation term.

Within the limitations discussed above and in Appendix A, we now set $\overline{\rho}_s'=0$. From Eqs. (8) and (14) we find the solution for the matrix elements of the single-electron density matrix to be

$$\rho'_{\mu\sigma,\nu\sigma'}=\frac{\mathcal{V}_{\mu\sigma,\nu\sigma'}}{i\tilde{\omega}\tau_p}\left[\Lambda^{(2)}_{\mu\sigma,\nu\sigma'}-(1-i\tilde{\omega}\tau_p)\Lambda^{(1)}_{\mu\sigma,\nu\sigma'}\right]+\frac{1-i\tilde{\omega}\tau_p}{i\tilde{\omega}\tau_p}\,n^*_{\nu\mu}\times\frac{[\Lambda^{(1)}_{\mu\sigma,\nu\sigma'}-\Lambda^{(2)}_{\mu\sigma,\nu\sigma'}]\,[K^{(1)}_{\sigma\sigma'}(n\overline{\mathcal{V}})-K^{(2)}_{\sigma\sigma'}(n\overline{\mathcal{V}})]}{K^{(1)}_{\sigma\sigma'}(n\overline{n})-(1-i\tilde{\omega}\tau_p)K^{(2)}_{\sigma\sigma'}(n\overline{n})}\ , \tag{15}$$

where $\tilde{\omega}=\omega-i/\tau$, and we have introduced the correlation functions

$$K^{(1,2)}_{\sigma\sigma'}(n\overline{n})=\frac{1}{V}\sum_{\mu\nu}|n_{\nu\mu}|^2\Lambda^{(1,2)}_{\mu\sigma,\nu\sigma'} \tag{16}$$

and

$$K^{(1,2)}_{\sigma\sigma'}(n\overline{\mathcal{V}})=\frac{1}{V}\sum_{\mu\nu}n_{\nu\mu}\mathcal{V}_{\mu\sigma,\nu\sigma'}\Lambda^{(1,2)}_{\mu\sigma,\nu\sigma'}\ . \tag{17}$$

Similar notation for various correlation functions will occur throughout the paper. The notation is meant to suggest, at a glance, which two operators are paired together inside the correlation function. From Eq. (15) one may compute various average values of physical quantities (e.g., the charge density) which develop in response to the perturbation $\mathcal{V}'(t)$. The ω Fourier component of the average value of an operator O is given by

$$\overline{O}(\omega)=\mathrm{tr}(O\rho')=\sum_{\substack{\mu\nu\\ \sigma\sigma'}}O_{\nu\sigma',\mu\sigma}\,\rho'_{\mu\sigma,\nu\sigma'}\ . \tag{18}$$

We mention here that Mertsching[12] has given a more general version of the simple RILE method, by introducing the collision term in the form

$$\left[\left(\frac{d\rho}{dt}\right)_c\right]_{\alpha\beta}=-\frac{1}{\tau_{\alpha\beta}}(\rho_{\alpha\beta}-\overline{\rho}_{\alpha\beta})\ .$$

This scheme would appear to be useful, e.g., for application to interband absorption processes, where different interband transitions might involve rather different relaxation times. Mertsching's modification can be incorporated into our scheme in a straightforward fashion, and has been omitted from our discussion, largely for simplicity.

Finally, while we have arrived at Eq. (14) in a somewhat *ad hoc* fashion, it should be emphasized that the procedure used is adequate for satisfying the physical conditions imposed in the Introduction. In addition, the method yields gauge-invariant results when applied to the general conductivity problem. In principle, one might be able to guess, at the outset, forms which satisfy these physical requirements, and we would view such an approach as legitimate. Furthermore our solution for $\bar{\rho}_p'$, $\bar{\rho}_s'$ is probably not unique. For example, one can readily construct a multitude of choices which satisfy charge conservation. The requirement of gauge invariance imposes additional constraints on the forms which can be chosen, but, again, the choice may not be unique. These features appear to be characteristic of relaxation-time ansatzes in both quantum and classical transport theory.

III. SIMPLE APPLICATIONS

A. Electrical Response

Using Eq. (15) we compute the current and charge densities of an electron gas which develop in response to a self-consistent electromagnetic field with vector and scalar potentials $\vec{\mathrm{A}}(\vec{\mathrm{q}}, \omega)$ and $\phi(\vec{\mathrm{q}}, \omega)$. Neglecting the interaction of the spin with the time-dependent magnetic field, the perturbation is

$$\mathcal{V} = -\frac{e}{c}\,\vec{\mathrm{j}}(-\vec{\mathrm{q}})\cdot\vec{\mathrm{A}}(\vec{\mathrm{q}}, \omega) + en(-\vec{\mathrm{q}})\,\phi(\vec{\mathrm{q}}, \omega)\;, \tag{19}$$

with

$$\vec{\mathrm{j}}(\vec{\mathrm{q}}) = \frac{1}{2m}\left[\left(\vec{\mathrm{p}} - \frac{e}{c}\vec{\mathrm{A}}_0\right)e^{i\vec{\mathrm{q}}\cdot\vec{\mathrm{r}}} + e^{i\vec{\mathrm{q}}\cdot\vec{\mathrm{r}}}\left(\vec{\mathrm{p}} - \frac{e}{c}\vec{\mathrm{A}}_0\right)\right]. \tag{20}$$

The induced charge and current densities are given by

$$n(\vec{\mathrm{q}}, \omega) = (e/V)\mathrm{tr}\big(n(\vec{\mathrm{q}})\rho'\big) \tag{21}$$

and

$$\vec{\mathrm{j}}(\vec{\mathrm{q}}, \omega) = (e/V)\mathrm{tr}\big(\vec{\mathrm{j}}(\vec{\mathrm{q}})\rho'\big) - (e^2/mc)\,\bar{N}\vec{\mathrm{A}}(\vec{\mathrm{q}}, \omega), \tag{22}$$

with $\bar{N} = \sum_\sigma \bar{N}_\sigma$ the electron density in equilibrium.

In this problem, no spin-spin or spin-charge correlation functions enter, and for this reason, we do not expect our results to be grossly different from those one obtains using the RILE method of Refs. 12–14. Nevertheless, we expect our method to yield a refinement, and because the results are not identical expect for zero magnetic field, we present them here. Also under the usual physical condition $\tau_s \gg \tau_p$, it is shown in Appendix A that the spin relaxation term plays a very minor role, and we set $1/\tau_s = 0$ in the subsequent discussion. We then find the following gauge-invariant results for the charge and current densities:

$$n(\vec{\mathrm{q}}, \omega)$$

$$= \frac{e^2}{i\omega}\sum_\sigma \frac{K_{\sigma\sigma}^{(1)}(n\bar{n})\vec{\mathrm{K}}_{\sigma\sigma}^{(2)}(n\bar{\mathrm{j}}) + i\omega\tau_p K_{\sigma\sigma}^{(2)}(n\bar{n})K_{\sigma\sigma}^{(1)}(n\bar{\mathrm{j}})}{K_{\sigma\sigma}^{(1)}(n\bar{n}) + i\omega\tau_p K_{\sigma\sigma}^{(2)}(n\bar{n})}\cdot\vec{\mathrm{E}}(\vec{\mathrm{q}}, \omega) \tag{23}$$

and

$$\vec{\mathrm{j}}(\vec{\mathrm{q}}, \omega) = \overleftrightarrow{\sigma}\cdot\vec{\mathrm{E}}(\vec{\mathrm{q}}, \omega), \tag{24}$$

where

$$\overleftrightarrow{\sigma} = \frac{e^2}{i\omega}\sum_\sigma\left(\frac{\bar{N}_\sigma}{m}\overleftrightarrow{\mathrm{I}} + \frac{i\omega\tau_p\overleftrightarrow{\mathrm{K}}_{\sigma\sigma}^{(1)}(\mathrm{j}\bar{\mathrm{j}}) + \overleftrightarrow{\mathrm{K}}_{\sigma\sigma}^{(2)}(\mathrm{j}\bar{\mathrm{j}})}{1 + i\omega\tau_p} - \frac{i\omega\tau_p}{1 + i\omega\tau_p}\;\frac{[\vec{\mathrm{K}}_{\sigma\sigma}^{(1)}(\mathrm{j}\bar{n}) - \vec{\mathrm{K}}_{\sigma\sigma}^{(2)}(\mathrm{j}\bar{n})][\vec{\mathrm{K}}_{\sigma\sigma}^{(1)}(n\bar{\mathrm{j}}) - \vec{\mathrm{K}}_{\sigma\sigma}^{(2)}(n\bar{\mathrm{j}})]}{K_{\sigma\sigma}^{(1)}(n\bar{n}) + i\omega\tau_p K_{\sigma\sigma}^{(2)}(n\bar{n})}\right) \tag{25}$$

is the conductivity tensor and $\overleftrightarrow{\mathrm{I}}$ is the unit tensor. In Eqs. (23) and (25) we have introduced the correlation functions

$$\vec{\mathrm{K}}_{\sigma\sigma}^{(1,2)}(\mathrm{j}\bar{n}) = \frac{1}{V}\sum_{\mu\nu}\vec{\mathrm{j}}_{\nu\mu}n_{\nu\mu}^{*}\Lambda_{\mu\sigma,\nu\sigma}^{(1,2)}\;, \tag{26}$$

$$\vec{\mathrm{K}}_{\sigma\sigma}^{(1,2)}(n\bar{\mathrm{j}}) = \frac{1}{V}\sum_{\mu\nu}n_{\nu\mu}\vec{\mathrm{j}}_{\nu\mu}^{*}\Lambda_{\mu\sigma,\nu\sigma}^{(1,2)}\;, \tag{27}$$

and

$$\overleftrightarrow{\mathrm{K}}_{\sigma\sigma}^{(1,2)}(\mathrm{j}\bar{\mathrm{j}}) = \frac{1}{V}\sum_{\mu\nu}\vec{\mathrm{j}}_{\nu\mu}\vec{\mathrm{j}}_{\nu\mu}^{*}\Lambda_{\mu\sigma,\nu\sigma}^{(1,2)}\;. \tag{28}$$

The gauge invariance of the results for $\vec{\mathrm{j}}$ and n follows from certain identities satisfied by the K correlation functions and given in Appendix B.

The dielectric tensor $\overleftrightarrow{\epsilon}$ is related to the conductivity by $\overleftrightarrow{\epsilon} = \overleftrightarrow{\mathrm{I}} + (4\pi/i\omega)\overleftrightarrow{\sigma}$. From the dielectric tensor one obtains the longitudinal dielectric function using the relation

$$\epsilon(\vec{\mathrm{q}}, \omega) = \vec{\mathrm{q}}\cdot\overleftrightarrow{\epsilon}\cdot\vec{\mathrm{q}}/q^2\;. \tag{29}$$

Using the identities in Appendix B, we find

$$\epsilon(\vec{\mathrm{q}}, \omega) = 1 - v_q\sum_\sigma\frac{(1 + i\omega\tau_p)K_{\sigma\sigma}^{(1)}(n\bar{n})K_{\sigma\sigma}^{(2)}(n\bar{n})}{K_{\sigma\sigma}^{(1)}(n\bar{n}) + i\omega\tau_p K_{\sigma\sigma}^{(2)}(n\bar{n})}\;, \tag{30}$$

where $v_q = 4\pi e^2/q^2$. The expression for $\epsilon(\vec{\mathrm{q}}, \omega)$ may readily be compared with the results of others. First, Eq. (30) may be compared to the results one obtains using the RILE method of Refs. 12–14:

$$\epsilon(\vec{q},\omega)=1-v_q\,\frac{(1+i\omega\tau_p)K^{(1)}(n\bar{n})K^{(2)}(n\bar{n})}{K^{(1)}(n\bar{n})+i\omega\tau_pK^{(2)}(n\bar{n})}\;, \tag{31}$$

where

$$K^{(1,2)}(n\bar{n})=\sum_{\sigma}K^{(1,2)}_{\sigma\sigma}(n\bar{n})\;.$$

In zero magnetic field Eqs. (30) and (31) are identical. It is also instructive to consider the zero-field quasiclassical limit of Eq. (30) for parabolic energy bands and a spherical Fermi surface:

$$\epsilon(\vec{q},\omega)=1-v_q\int d^3\vec{v}\,\frac{df}{d\epsilon}\,\frac{\vec{q}\cdot\vec{v}}{\tilde{\omega}+\vec{q}\cdot\vec{v}}\Bigg/\left(1+\frac{i}{\tau_p}\left\langle\!\!\left\langle\frac{1}{\tilde{\omega}+\vec{q}\cdot\vec{v}}\right\rangle\!\!\right\rangle\right), \tag{32}$$

where $\langle\langle A\rangle\rangle$ denotes the average value of A defined by

$$\langle\langle A\rangle\rangle=\int d^3\vec{v}\,\frac{df}{d\epsilon}\,A\Bigg/\int d^3\vec{v}\,\frac{df}{d\epsilon}\;,$$

and $f=2(m/2\pi\hbar)^3\rho_0$ is the classical equilibrium distribution function. This is identical to the result one obtains using the well-known procedure of Cohen, Harrison, and Harrison.[9] A similar result was obtained by Hamilton and McWhorter[24] using an ansatz[10] which conserves charge, spin, and kinetic energy. Their result may be written

$$\epsilon(\vec{q},\omega)=1-v_q\int d^3\vec{v}\left[\frac{df}{d\epsilon}\left(\frac{\vec{q}\cdot\vec{v}}{\tilde{\omega}+\vec{q}\cdot\vec{v}}\right)\Bigg/\left(1+\frac{i}{\tau_p}\left\langle\frac{1}{\tilde{\omega}+\vec{q}\cdot\vec{v}}\right\rangle\right)\right], \tag{33}$$

where the angular brackets denote average over angles in momentum space. Equations (32) and (33) are quite similar and become identical for a zero-temperature Fermi gas.

B. Magnetic Response

Here we compute the spin magnetization of an electron gas which develops in response to a magnetic field $\vec{B}(\vec{r},t)=\vec{B}(\vec{q},\omega)\,e^{i(\omega t-\vec{q}\cdot\vec{r})}$. The perturbation $\mathcal{V}$ is

$$\mathcal{V}=(\tfrac{1}{2}g\mu_B)\,n(-\vec{q})\vec{\sigma}\cdot\vec{B}(\vec{q},\omega)\;, \tag{34}$$

where $\frac{1}{2}g\mu_B$ is the electron magnetic moment with μ_B the Bohr magneton. In this problem it is crucial to include τ_s in Eq. (15) even though $\tau_s\gg\tau_p$. The reason is that with τ_p a spin-conserving relaxation process, we shall find, under conditions of long-wavelength spin resonance, that τ_p *cancels out* leaving *only* τ_s to broaden the resonance, as would be expected physically.

The induced magnetization is given by

$$\vec{m}(\vec{q},\omega)=(\tfrac{1}{2}g\mu_B)(1/V)\,\mathrm{tr}[\,\vec{\sigma}n(\vec{q})\rho'\,]\;. \tag{35}$$

Using Eq. (15) and defining the spin susceptibility tensor $\overleftrightarrow{S}$ by $\vec{m}=\overleftrightarrow{S}\cdot\vec{B}(\vec{q},\omega)$, we find[25]

$$\overleftrightarrow{S}=(\tfrac{1}{2}g\mu_B)^2\sum_{\sigma\sigma'}\langle\sigma'|\vec{\sigma}|\sigma\rangle\langle\sigma|\vec{\sigma}|\sigma'\rangle G_{\sigma,\sigma'}\;, \tag{36}$$

where

$$G_{\sigma,\sigma'}=\frac{i\tilde{\omega}\tau_pK^{(1)}_{\sigma\sigma'}(n\bar{n})K^{(2)}_{\sigma\sigma'}(n\bar{n})}{K^{(1)}_{\sigma\sigma'}(n\bar{n})-(1-i\tilde{\omega}\tau_p)K^{(2)}_{\sigma\sigma'}(n\bar{n})}\;. \tag{37}$$

In the long-wavelength limit ($\vec{q}\to 0$), the resonant magnetization response to an oscillating $\vec{B}$ field which is perpendicular to $\vec{B}_0$ is related to the perpendicular susceptibility component $S_+=S_{xx}+iS_{xy}$. Using Eqs. (36) and (37) one can show that

$$S_+=2\delta N(\tfrac{1}{2}g\mu_B)^2\left(\frac{1}{\hbar(\omega-\omega_s)-i\hbar/\tau_s}\right)\;. \tag{38}$$

Here, $\hbar\omega_s=\epsilon_{\mu\uparrow}-\epsilon_{\mu\downarrow}=\epsilon_\uparrow-\epsilon_\downarrow$ is the spin-flip energy (we assume $\hbar\omega_s>0$), and $\delta N=\bar{N}_\downarrow-\bar{N}_\uparrow$ is the difference in spin populations. We see that, in the present treatment, the spin resonance is broadened only by $1/\tau_s$, and the quantity $1/\tau_p$ does not occur in Eq. (38). If we had used the simple RILE method, which does not conserve spin density for orbital relaxation processes, we would find the spin resonance broadened by $1/\tau\simeq 1/\tau_p\gg 1/\tau_s$. This result is clearly unphysical. Finally, while the resonant response is correctly given by Eqs. (36) and (37), the nonresonant components (S_- for $\omega>0$ or S_+ for $\omega<0$) yield unphysical results.[3,4,6,7] This is due to the approximation $(\bar{\rho}_s')=0$, and may be corrected using the full relaxation-time ansatz where $(\bar{\rho}_s')_{\mu\sigma,\nu\sigma'}$ is given by Eq. (13) (see Appendix A).

IV. APPLICATION TO INELASTIC LIGHT SCATTERING FROM SEMICONDUCTOR MAGNETOPLASMAS

A theoretical discussion of light scattering from semiconductor plasmas in a magnetic field has been given in Ref. 20. Relaxation effects were neglected in this treatment. Here we shall generalize the results of Ref. 20 to include damping effects using the ansatz of Sec. II. We shall follow the notation of this paper where it is convenient. Following Ref. 20, we adopt the approximation of Hamilton and McWhorter[24] which neglects electro-optic and magneto-optic coupling, and we treat the remaining Coulomb interaction in the random-phase approximation (RPA).

In this approximation, the differential cross section for the scattering of a photon from the state $(\omega_I,\vec{k}_I,\vec{\epsilon}_I)$ to the state $(\omega_F,\vec{k}_F,\vec{\epsilon}_F)$ and the concomitant transition of the many-electron system from state $|I\rangle$ to state $|F\rangle$ is given by

$$\frac{d^2\sigma}{d\Omega d\omega}=\hbar\frac{\omega_F}{\omega_I}\,\sigma_T\left\langle \sum_F |M_{FI}|^2\,\delta(E_F-E_I-\hbar\omega)\right\rangle , \tag{39}$$

where $\sigma_T=(e^2/mc^2)^2$ is the Thomson cross section, and $\langle\cdots\rangle$ denotes a statistical average over the initial equilibrium distribution of the many-electron system. Also, M_{FI} is the matrix element of the pair operator N,

$$M_{FI}=\langle F|N|I\rangle=\left\langle F\left|\sum_{\alpha\beta}\gamma_{\alpha\beta}c_\alpha^\dagger c_\beta\right|I\right\rangle . \tag{40}$$

Here $c_\alpha^\dagger$ and c_β are the creation and destruction operators for the eigenstates $|\alpha\rangle$ and $|\beta\rangle$ of the one-electron Hamiltonian H_0. We define $\omega=(\omega_I-\omega_F)$, and $\vec{q}=(\vec{k}_I-\vec{k}_F)$. The matrix element $\gamma_{\alpha\beta}$ is given by

$$\gamma_{\alpha\beta}=n_{\alpha\beta}\vec{\epsilon}_I\cdot\vec{\epsilon}_F+\frac{1}{m}\sum_{\beta'}\left[\frac{\langle\alpha|j_F|\beta'\rangle\langle\beta'|j_I|\beta\rangle}{\epsilon_\beta-\epsilon_{\beta'}+\hbar\omega_I}+\frac{\langle\alpha|j_I|\beta'\rangle\langle\beta'|j_F|\beta\rangle}{\epsilon_\alpha-\epsilon_{\beta'}-\hbar\omega_I}\right] , \tag{41}$$

where $n_{\alpha\beta}$ is given by Eqs. (10), and

$$\begin{Bmatrix} j_I \\ j_F\end{Bmatrix}=\begin{Bmatrix}\vec{\epsilon}_I\\ \vec{\epsilon}_F\end{Bmatrix}\cdot\left(\vec{\pi}-\frac{e}{c}\vec{A}_0\right)\exp\left[i\begin{Bmatrix}\vec{k}_I\\ -\vec{k}_F\end{Bmatrix}\cdot\vec{r}\right] , \tag{42}$$

with $\vec{\pi}=\vec{p}+\hbar(\vec{\sigma}\times\vec{\nabla}V)/(4mc^2)$ and $\vec{A}_0$ the vector potential for the uniform magnetic field.

In order to make use of the results derived in Sec. II, it is convenient for our purposes to proceed with the calculation of the cross section using a density matrix formalism, rather than the Green's-function approach employed in Ref. 20. It can be shown that the cross-section expression may be rewritten in the form

$$\frac{d^2\sigma}{d\Omega d\omega}=\hbar\frac{\omega_F}{\omega_I}\sigma_T\frac{V}{v_q}\;\frac{(n_\omega+1)}{\pi}\;\mathrm{Im}[\mathrm{Tr}(RN)] , \tag{43}$$

where $n_\omega=(e^{\hbar\omega/kT}-1)^{-1}$ is the Bose function, $v_q=4\pi e^2/q^2$, and V is the scattering volume. Also in this equation, R is the Fourier component of the linear-response part of the many-electron density matrix to a time-dependent perturbation of the form

$$\mathcal{V}'(t)=(v_q/V)N^\dagger e^{i(\omega-i0^+)t} , \tag{44}$$

where $N^\dagger$ is the Hermitian conjugate of N defined in Eq. (40). That is, R is the solution of the equation of motion

$$(-\hbar\omega+i0^+)R=[H,R]+(v_q/V)[N^\dagger,\rho_0] , \tag{45}$$

where ρ_0 is the thermal-equilibrium canonical density matrix, and H is the full many-electron Hamiltonian including Coulomb interactions. Evaluation of Eqs. (43) and (45) in the representation of exact eigenstates of H and ρ_0 immediately yields the original cross-section formula [Eq. (39)]. In effect, we have replaced the cross-section calculation by an equivalent linear-transport problem.

Since N is just a one-electron operator, it is convenient to introduce the single-particle density matrix whose matrix elements in the one-electron representation[26] are given by

$$\rho'_{\alpha\beta}=\mathrm{Tr}(c_\beta^\dagger c_\alpha R) . \tag{46}$$

Then, we have

$$I(\omega)=\mathrm{tr}\;(NR)=\mathrm{tr}(\gamma\rho') , \tag{47}$$

where tr denotes trace over one-electron states, and γ is the operator whose matrix elements are given by Eq. (41). Due to the Coulomb term in the full Hamiltonian H, the single-particle density matrix is coupled to a hierarchy of higher-order density matrices. Decoupling in the RPA, we obtain the equation of motion

$$(-\hbar\omega)\rho'=[H_0,\rho']+[\mathcal{V}_{\mathrm{tot}},\rho_0]+i\hbar\left(\frac{d\rho'}{dt}\right)_c , \tag{48}$$

where the phenomenological relaxation term $(d\rho'/dt)_c$ has been inserted, and $\mathrm{V}_{\mathrm{tot}}=\mathrm{V}_{\mathrm{ex}}+\mathrm{V}_{\mathrm{ind}}$ is the sum of an external and an induced perturbation:

$$\mathcal{V}_{\mathrm{ex}}=(v_q/V)\gamma^\dagger ,$$

and

$$\mathcal{V}_{\mathrm{ind}}=(v_q/V)n(-\vec{q})\,\mathrm{tr}[n(\vec{q})\rho'] .$$

Note that Eq. (48) is identical in form with Eq. (8) of Sec. II. Making use of Eq. (15) with $\mathcal{V}$ replaced by $\mathcal{V}_{\mathrm{tot}}$, we find from Eqs. (43) and (47) the result

$$\frac{d^2\sigma}{d\Omega d\omega}=\sigma_T\hbar\frac{\omega_F}{\omega_I}\left(\frac{n_\omega+1}{\pi}\right)\frac{V}{v_q}\,\mathrm{Im}I(\omega) ,$$

where

$$I(\omega)=\frac{-v_q}{i\tilde{\omega}\tau_p}\sum_{\sigma\sigma'}\left([(1-i\tilde{\omega}\tau_p)K^{(1)}_{\sigma\sigma'}(\gamma\bar{\gamma})-K^{(2)}_{\sigma\sigma'}(\gamma\bar{\gamma})]-(1-i\tilde{\omega}\tau_p)\frac{[K^{(1)}_{\sigma\sigma'}(n\bar{\gamma})-K^{(2)}_{\sigma\sigma'}(n\bar{\gamma})][K^{(1)}_{\sigma\sigma'}(\gamma\bar{n})-K^{(2)}_{\sigma\sigma'}(\gamma\bar{n})]}{K^{(1)}_{\sigma\sigma'}(n\bar{n})-(1-i\tilde{\omega}\tau_p)K^{(2)}_{\sigma\sigma'}(n\bar{n})}\right)$$

$$+\frac{v_q^2}{\epsilon(\vec{q},\omega)}\sum_\sigma\frac{K^{(1)}_{\sigma\sigma}(n\bar{n})K^{(2)}_{\sigma\sigma}(n\bar{\gamma})-(1-i\tilde{\omega}\tau_p)K^{(2)}_{\sigma\sigma}(n\bar{n})K^{(1)}_{\sigma\sigma}(n\bar{\gamma})}{K^{(1)}_{\sigma\sigma}(n\bar{n})-(1-i\tilde{\omega}\tau_p)K^{(2)}_{\sigma\sigma}(n\bar{n})}\sum_\sigma\frac{K^{(1)}_{\sigma\sigma}(n\bar{n})K^{(2)}_{\sigma\sigma}(\gamma\bar{n})-(1-i\tilde{\omega}\tau_p)K^{(2)}_{\sigma\sigma}(n\bar{n})K^{(1)}_{\sigma\sigma}(\gamma\bar{n})}{K^{(1)}_{\sigma\sigma}(n\bar{n})-(1-i\tilde{\omega}\tau_p)K^{(2)}_{\sigma\sigma}(n\bar{n})} . \tag{49}$$

In Eq. (49) we have defined

$$K^{(1,2)}_{\sigma\sigma'}(\gamma\overline{\gamma})=\frac{1}{V}\sum_{\mu\nu}\gamma_{\nu\sigma',\mu\sigma}\gamma^{*}_{\nu\sigma',\mu\sigma}\Lambda^{(1,2)}_{\mu\sigma,\nu\sigma'}\ , \tag{50}$$

$$K^{(1,2)}_{\sigma\sigma'}(\gamma\overline{n})=\frac{1}{V}\sum_{\mu\nu}\gamma_{\nu\sigma',\mu\sigma}n^{*}_{\nu\mu}\Lambda^{(1,2)}_{\mu\sigma,\nu\sigma'}\ , \tag{51}$$

$$K^{(1,2)}_{\sigma\sigma'}(n\overline{\gamma})=\frac{1}{V}\sum_{\mu\nu}n_{\nu\mu}\gamma^{*}_{\nu\sigma',\mu\sigma}\Lambda^{(1,2)}_{\mu\sigma,\nu\sigma'}\ , \tag{52}$$

and ϵ is the longitudinal dielectric function

$$\epsilon(\vec{q},\ \omega)=1-v_q\textstyle\sum_\sigma G_{\sigma,\sigma}\ , \tag{53}$$

where $G_{\sigma,\sigma}$ is given by Eq. (37). For $1/\tau_s=0$ this becomes identical to Eq. (30) of Sec. III A.

Several remarks concerning Eq. (49) are in order. In the infinite relaxation-time approximation ($1/\tau=0$), it reduces[27] to the result given previously [see Eq. (11) of Ref. 20]. For simplicity we have neglected the phonon contributions to ϵ. We have derived Eq. (49) in the spirit of interest in scattering from intraband excitations within one energy band (e.g., the conduction band) which is not strongly spin-orbit mixed. This does not exclude spin-dependent scattering processes (e.g., spin flip) from our treatment. Such processes may result[24,28,29] from virtual interband contributions to $\gamma_{\alpha\beta}$ and are correctly treated by our formalism. Finally, the usefulness of our ansatz is not tied to the approximate light-scattering calculation of Ref. 20. It can be applied to any physically appropriate situation which can be formulated in terms of the one-electron density matrix theory as described above.

While the general result in Eq. (49) is rather complicated in appearance, it reduces considerably in certain special cases. The complexity of Eq. (49) derives largely from the structure of the $\gamma_{\alpha\beta}$ matrix elements. These matrix elements may be approximated under certain circumstances, but the approximations one can make depend on the physical situation and the scattering process being considered. A detailed application of Eq. (49) to quasi-elastic scattering from acoustic plasma waves and single-electron excitations in semiconductor magnetoplasmas will be given in a subsequent paper.[22] For illustration of the general utility of our result, we now consider two simple examples. First, we consider an effective-mass approximation, which is roughly appropriate for the case where the incident photon energy is small compared to any characteristic energy band gaps. The effect of interband terms in γ is to produce an effective-mass correction so that

$$\gamma_{\nu\sigma',\mu\sigma}\approx(m/m^{*})\vec{\epsilon}_I\cdot\vec{\epsilon}_F\,n_{\nu\mu}\,\delta_{\sigma,\sigma'}\ ,$$

and Eq. (49) collapses to give

$$\frac{d^2\sigma}{d\Omega d\omega}=\sigma^{*}_{T}\hbar\,\frac{\omega_F}{\omega_I}\left(\frac{n_\omega+1}{\pi}\right)\frac{v}{v_q}(\vec{\epsilon}_I\cdot\vec{\epsilon}_F)^2\times\mathrm{Im}\left(\frac{1}{\epsilon(\vec{q},\ \omega)}\right). \tag{54}$$

This result is equivalent to those derived using a one-band effective-mass Hamiltonian.[30] Relaxation effects appear only through the longitudinal dielectric function. As a second example, we consider conduction-band spin-flip excitations in the III-V semiconductors.[28,29] For ω near the spin-flip frequency ω_s, we include only spin-flip contributions to γ which have the form[28,29]

$$\gamma_{\nu\sigma',\mu\sigma}\simeq iB\left|(\vec{\epsilon}_I\times\vec{\epsilon}_F)_\perp\right|n_{\nu\mu}\,\delta_{\sigma',-\sigma}\ , \tag{55}$$

where B is a constant and the subscript $\perp$ denotes the component of the vector perpendicular to $\vec{B}_0$. The contribution of Eq. (55) to the cross section is readily found to be

$$\frac{d^2\sigma}{d\Omega d\omega}=\sigma_T B^2\hbar\,\frac{\omega_F}{\omega_I}\left(\frac{n_\omega+1}{\pi}\right)V\left|(\vec{\epsilon}_I\times\vec{\epsilon}_F)_\perp\right|^2\times\mathrm{Im}\sum_\sigma G_{\sigma,-\sigma}\ , \tag{56}$$

where $G_{\sigma,\sigma'}$ is given by Eq. (37). It follows from the discussion of Sec. III B that, for long wavelengths and a material in which the spin-flip energy is independent of the orbital quantum numbers, the spin-flip scattering will be broadened only by the spin relaxation (τ_s).

Perhaps the greatest utility of Eq. (49) lies not in simple examples such as these, but in treating more complex situations such as the magnetoacoustic plasma waves being driven by spin density oscillations,[22] or situations in which band nonparabolicity and/or finite-wavelength effects enhance the competition between orbital and spin relaxation. The acoustic plasma-wave problem is an interesting case in which τ_s is relatively unimportant and yet it is essential to use a spin-conserving ansatz for the orbital (τ_p) term. This results because, although no spin-flip processes are involved, the important contribution to the cross section arises from spin density oscillations (m_z). We would also caution that there may be situations of practical interest in which it is necessary to use the more general density matrix results given in Appendix A to obtain the cross section, rather than the above results based on the approximation $\overline{\rho}_s=\rho_0(H_0,\ \mu_0)$.

V. DISCUSSION

We have presented a phenomenological relaxation-time ansatz which encompasses both orbital and spin relaxation. Our ansatz conserves spin density for orbital relaxation processes, and we have shown the importance of satisfying this requirement for certain physical problems. The results are reasonably general, and should find application in other areas. Some caution should be exercised, however, in applying the results of this paper to other cases. The physical requirements of what constitutes a reasonable relaxation-time ansatz can vary from problem to problem. A case in point might involve the application of these results to the calculation of something like the thermal conductivity. Here considerations of energy conservation will be important, and our ansatz is not explicitly designed to properly treat energy transport. Another class of problems where our method cannot be applied directly is that in which spin-orbit coupling enters in a more direct way. Of course, the whole question of what one means by orbital and spin relaxation becomes tenuous if strong spin-orbit coupling is involved.

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APPENDIX A

Here we present the complete density matrix solution which obtains using Eqs. (13) and (14) in the collision terms, and we discuss some results which obtain from this solution. First, we define the quantities

$$D_{\sigma\sigma'} = K^{(1)}_{\sigma\sigma'}(n\bar{n}) - (1 - i\tilde{\omega}\tau_p)K^{(2)}_{\sigma\sigma'}(n\bar{n}) \,, \tag{A1}$$

$$F_{\sigma\sigma'} = (1/D_{\sigma\sigma'})[K^{(1)}_{\sigma\sigma'}(n\bar{n})K^{(2)}_{\sigma\sigma'}(n\bar{\mathfrak{v}}) + i\omega\tau_p K^{(2)}_{\sigma\sigma'}(n\bar{n})\,K^{(1)}_{\sigma\sigma'}(n\bar{\mathfrak{v}})] \,, \tag{A2}$$

$$J_{\sigma\sigma'} = (1/D_{\sigma\sigma'})K^{(2)}_{\sigma\sigma'}(n\bar{n})K^{(2)}_{\sigma\sigma'}(n\bar{\mathfrak{v}}) \,, \tag{A3}$$

$$H_{\sigma\sigma'} = (1/D_{\sigma\sigma'})K^{(2)}_{\sigma\sigma'}(n\bar{n})[K^{(1)}_{\sigma\sigma'}(n\bar{n}) - K^{(2)}_{\sigma\sigma'}(n\bar{n})] \,, \tag{A4}$$

$$K^{(1,2)}(n\bar{\mathfrak{v}}) = \sum_{\sigma} K^{(1,2)}_{\sigma\sigma}(n\bar{\mathfrak{v}}) \,, \tag{A5}$$

$$K^{(1,2)}(n\bar{n}) = \sum_{\sigma} K^{(1,2)}_{\sigma\sigma}(n\bar{n}) \,. \tag{A6}$$

Also we define

$$Q_{\sigma\sigma'} = \frac{1}{V}\sum_{\mu\nu} n_{\nu\mu}\rho'_{\mu\sigma,\nu\sigma'} \,. \tag{A7}$$

The solution for $Q_{\sigma\sigma'}$ is then

$$Q_{\sigma\sigma'} = F_{\sigma\sigma'} + \frac{\tau_p}{\tau_s}\left(J_{\sigma\sigma'} - \frac{Q - K^{(2)}(n\bar{\mathfrak{v}})}{K^{(2)}(n\bar{n})} H_{\sigma\sigma}\delta_{\sigma,\sigma'}\right) \,, \tag{A8}$$

where the solution for $Q = \sum_\sigma Q_{\sigma\sigma}$ is seen to be

$$Q = \sum\nolimits_\sigma \left[F_{\sigma\sigma} + \frac{\tau_p}{\tau_s}\left(J_{\sigma\sigma} + \frac{K^{(2)}(n\bar{\mathfrak{v}})}{K^{(2)}(n\bar{n})} H_{\sigma\sigma}\right)\right] \Bigg/ \left(1 + \frac{\tau_p}{\tau_s}\frac{1}{K^{(2)}(n\bar{n})}\sum_{\sigma} H_{\sigma\sigma}\right) . \tag{A9}$$

The solution for the density matrix is given by

$$\begin{aligned}\rho'_{\mu\sigma,\nu\sigma'} &= \mathfrak{v}_{\mu\sigma,\nu\sigma'}\Lambda^{(1)}_{\mu\sigma,\nu\sigma'} \\ &+ \frac{i}{\tilde{\omega}\tau}\mathfrak{v}_{\mu\sigma,\nu\sigma'}(\Lambda^{(1)}_{\mu\sigma,\nu\sigma'} - \Lambda^{(2)}_{\mu\sigma,\nu\sigma'}) \\ &+ \frac{i}{\tilde{\omega}\tau_p} n^*_{\nu\mu}(\Lambda^{(1)}_{\mu\sigma,\nu\sigma'} - \Lambda^{(2)}_{\mu\sigma,\nu\sigma'})\frac{Q_{\sigma\sigma'} - K^{(2)}_{\sigma\sigma'}(n\bar{\mathfrak{v}})}{K^{(2)}_{\sigma\sigma'}(n\bar{n})} \\ &+ \frac{i}{\tilde{\omega}\tau_s} n^*_{\nu\mu}(\Lambda^{(1)}_{\mu\sigma,\nu\sigma} - \Lambda^{(2)}_{\mu\sigma,\nu\sigma})\delta_{\sigma,\sigma'}\frac{Q - K^{(2)}(n\bar{\mathfrak{v}})}{K^{(2)}(n\bar{n})} \,.\end{aligned} \tag{A10}$$

Applied to the general conductivity problem, Eq. (A10) may be shown to yield gauge-invariant charge-conserving results for the induced current and charge densities. The result for the longitudinal dielectric function from this calculation is

$$\epsilon(\vec{q}, \omega) = 1 - v_q \sum\nolimits_\sigma G_{\sigma,\sigma}/D \,, \tag{A11}$$

where $G_{\sigma,\sigma'}$ is given by Eq. (37), and

$$D = \left\{1 + \frac{\tau_p}{\tau_s}\frac{1}{K^{(2)}(n\bar{n})}\sum_{\sigma} H_{\sigma\sigma}\right\} . \tag{A12}$$

Equation (A11) has several interesting limits. For $\tau_s \gg \tau_p$ (without restriction on the value of $\omega\tau_s$), it reduces to yield Eq. (30) of Sec. III A. Another interesting (but probably unphysical) limit of (A11) is the case $1/\tau \to 0$ with $1/\tau_s$ finite. In this case Eq. (A11) reduces to give the simple RILE result [Eq. (31)] with τ_p replaced by τ_s. It is seen from these results that τ_s plays a minor role in the conductivity problem as long as the condition $\tau_s \gg \tau_p$ is satisfied.

Application of Eq. (A10) to the long-wavelength perpendicular magnetization response discussed in Sec. III B leads to the following results for $S_\pm = S_{xx} \pm iS_{xy}$:

$$S_\pm = \pm 2\,\delta N\,\frac{(\frac{1}{2}g\mu_B)^2(1 \pm i/\omega_s\tau_s)}{\hbar(\omega \mp \omega_s) - i\hbar/\tau_s} \,. \tag{A13}$$

These results differ from those which one obtains from the analysis in Sec. III B by the factors $(1 \pm i/\omega_s\tau_s)$ in the numerator of (A13). This factor was unity previously. The power absorbed from the $S_\pm$ components of the response is proportional to $\omega\,\mathrm{Im}S_\pm$. From Eq. (A13) we obtain

$$\omega\,\mathrm{Im}S_\pm = \frac{2\,\delta N(\frac{1}{2}g\mu_B)^2(\omega^2/\omega_s)(\hbar/\tau_s)}{\hbar^2(\omega \mp \omega_s)^2 + \hbar^2/\tau_s^2} \,, \tag{A14}$$

which is always positive. Our previous treatment, which is equivalent to spin relaxation to the static equilibrium state, would give

$$\omega\,\mathrm{Im}S_{\pm} = \pm\frac{2\delta N(\frac{1}{2}g\mu_B)^2\omega(\hbar/\tau_s)}{\hbar^2(\omega\mp\omega_s)^2+\hbar^2/\tau_s^2}\ . \tag{A15}$$

At resonance ($\omega=\omega_s$ for the S_+ component, $\omega=-\omega_s$ for the S_- component) Eqs. (A14) and (A15) yield identical results. However the nonresonant response given by Eq. (A15) has the wrong sign and is unphysical. This difficulty is well known from the theory of molecular resonance (see, e.g., Ref. 4), and is resolved by allowing the collision processes to relax the system to an equilibrium state determined by the instantaneous Hamiltonian.

APPENDIX B

The gauge invariance of the results discussed in Sec. III A and in Appendix A obtains from the following identities:

$$\vec{\mathrm{q}}\cdot\overleftrightarrow{\mathrm{K}}^{(1)}_{\sigma\sigma}(\mathrm{j}\bar{\mathrm{j}}) = -(\vec{\mathrm{q}}/m)\bar{N}_\sigma + \tilde{\omega}\vec{\mathrm{K}}^{(1)}_{\sigma\sigma}(n\bar{\mathrm{j}})\ , \tag{B1}$$

$$\overleftrightarrow{\mathrm{K}}^{(1)}_{\sigma\sigma}(\mathrm{j}\bar{\mathrm{j}})\cdot\vec{\mathrm{q}} = -(\vec{\mathrm{q}}/m)\bar{N}_\sigma + \tilde{\omega}\vec{\mathrm{K}}^{(1)}_{\sigma\sigma}(\mathrm{j}\bar{n})\ , \tag{B2}$$

$$\vec{\mathrm{q}}\cdot\overleftrightarrow{\mathrm{K}}^{(2)}_{\sigma\sigma}(\mathrm{j}\bar{\mathrm{j}}) = \overleftrightarrow{\mathrm{K}}^{(2)}_{\sigma\sigma}(\mathrm{j}\bar{\mathrm{j}})\cdot\vec{\mathrm{q}} = -\frac{\vec{\mathrm{q}}}{m}\bar{N}_\sigma\ , \tag{B3}$$

$$\vec{\mathrm{q}}\cdot\vec{\mathrm{K}}^{(1)}_{\sigma\sigma}(\mathrm{j}\bar{n}) = \vec{\mathrm{K}}^{(1)}_{\sigma\sigma}(n\bar{\mathrm{j}})\cdot\vec{\mathrm{q}} = \tilde{\omega}K^{(1)}_{\sigma\sigma}(n\bar{n})\ , \tag{B4}$$

and

$$\vec{\mathrm{q}}\cdot\vec{\mathrm{K}}^{(2)}_{\sigma\sigma}(\mathrm{j}\bar{n}) = \vec{\mathrm{K}}^{(2)}_{\sigma\sigma}(n\bar{\mathrm{j}})\cdot\vec{\mathrm{q}} = 0\ . \tag{B5}$$

All of these identities follow from the definitions Eqs. (16), (26)–(28), and from various standard operator identities, e.g.,

$$[H_0,\, n(\vec{\mathrm{q}})] = \hbar\vec{\mathrm{q}}\cdot\vec{\mathrm{j}}(\vec{\mathrm{q}})\ , \tag{B6}$$

$$[\vec{\mathrm{j}}(-\vec{\mathrm{q}}),\, n(\vec{\mathrm{q}})] = \hbar\vec{\mathrm{q}}/m\ . \tag{B7}$$

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